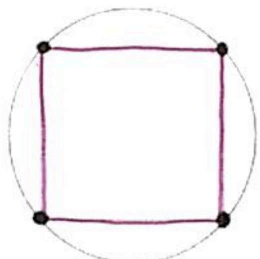
PAP Geometry Notes
Chapter 12 - Circles

Name: Coach C's Notes

12.1:
Inscribed and Circumscribed Triangles and Quadrilaterals

- **Inscribed polygon**: a polygon drawn inside a circle such that each vertex of the polygon touches the circle. (inscribed → drawn inside)
- **Circumscribed polygon**: a polygon drawn outside a circle such that each side of the polygon is tangent to the circle. (circumscribed → drawn around)

inscribed triangle

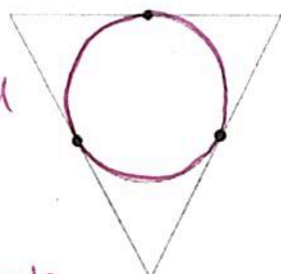


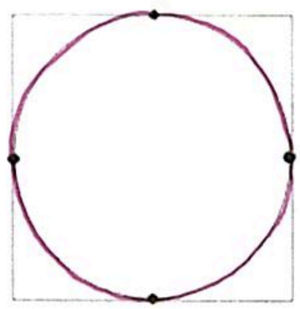
Inscribed Square

Circumscribed triangle

or

inscribed circle





Circumscribed Square

or

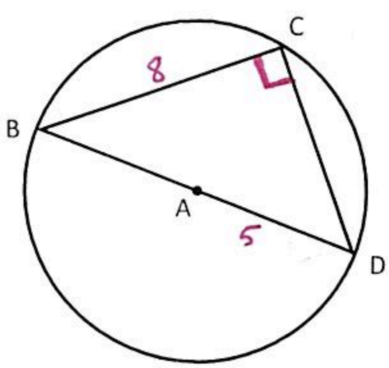
inscribed circle

The **Inscribed Right Triangle-Diameter theorem** states that if a triangle is inscribed in a circle such that one side of the triangle is a diameter of the circle, then the triangle is a right triangle.

$\overline{AD} = 5$

$\overline{BC} = 8$

Find $m\overline{CD}$



$\overline{BD} = 10$

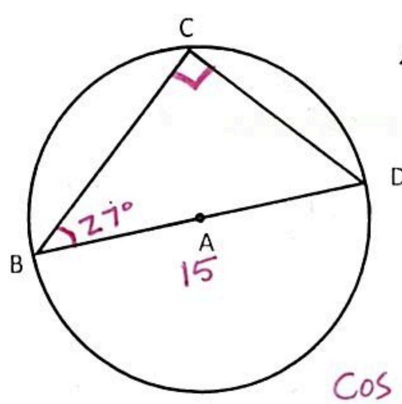
$8^2 + \overline{CD}^2 = 10^2$

$\overline{CD} = 6$

$\overline{BD} = 15$

$\angle DBC = 27°$

Find $m\overline{BC}$



SOH
CAH
TOA

$\cos 27° = \dfrac{\overline{BC}}{15}$

$(15)\cos 27° = \overline{BC}$

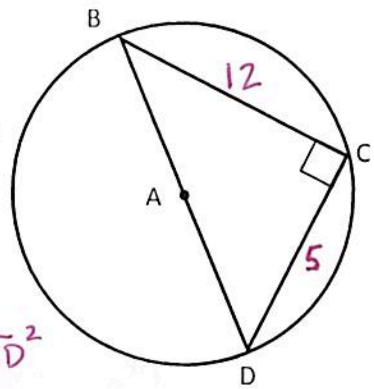
$\overline{BC} = 13.37$

The **Inscribed Right Triangle-Diameter *Converse* theorem** states that if a right triangle is inscribed in a circle, then the hypotenuse is a diameter of the circle.

$\overline{CD} = 5$

$\overline{BC} = 12$

Find the radius



$12^2 + 5^2 = \overline{BD}^2$

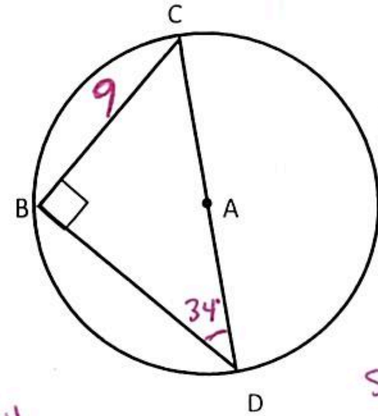
$\overline{BD} = 13$

radius = 6.5

$\overline{CB} = 9$

$\angle CDB = 34°$

Find the diameter.



SOH
CAH
TOA

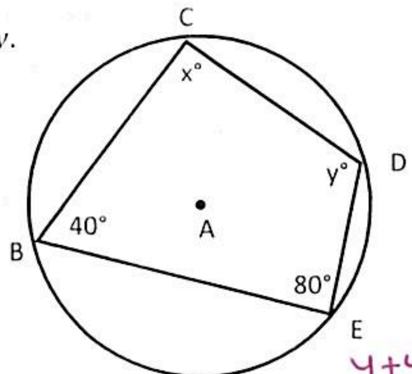
$\sin 34° = \dfrac{9}{\overline{CD}}$

$\overline{CD} = \dfrac{9}{\sin 34}$

diameter $\overline{CD} = 16.09$

The **Inscribed Quadrilateral-Opposite Angles theorem** states that if a quadrilateral is inscribed in a circle, then the opposite angles are supplementary.

Find x and y.



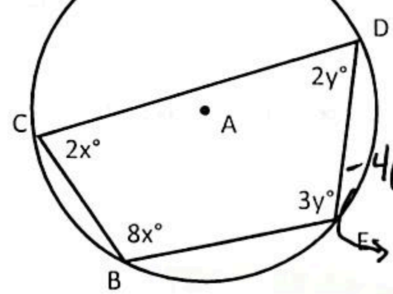
$x + 80 = 180$

$x = 100$

$y + 40 = 180$

$y = 140$

Find x and y.



$-4(2x + 3y = 180)$

$8x + 2y = 180$

$-8x - 12y = -720$

$-10y = -540$

$y = 54$

$2x + 3(54) = 180$

$2x + 162 = 180$

$x = 9$

Check for Understanding:

$\overleftrightarrow{GB}$ is a diameter of circle O. $\overleftrightarrow{AD}$ is tangent to circle O at point A. $m\angle GBA = 38°$.

→ right Δ

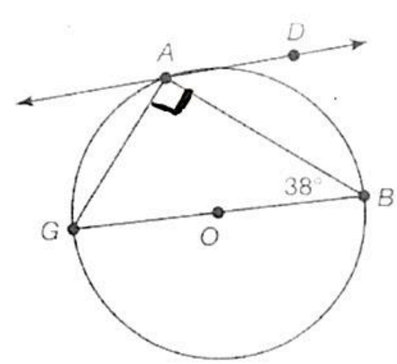
Find:

$m\angle GAB$ 90°

$m\angle G$ 52°

$m\,\widehat{AG}$ 76°

$m\,\widehat{AB}$ 104°



12.2: Arc Length

- **Arc Length**: a portion of the circumference of a circle. The length of an arc is different from the degree measure of the arc. Arcs are measured in degrees whereas arc lengths are linear measurements.

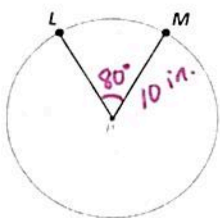
There is a proportional relationship between the measure of an arc length of a circle, s, and the circumference of the circle. To measure arc length, s, you multiply the circumference of the circle by a fraction that represents the portion of the circumference determined by the central angle measure, m.

$$s = \frac{m}{360^\circ} \cdot circumference$$

$$s = \frac{m}{360^\circ} \cdot (2\pi r)$$

$s =$ arc length
$m =$ Central $\measuredangle$
$r =$ radius

Example 1: Determine the measure of an arc for a circle with a radius of 10 inches and a central angle of 80°.

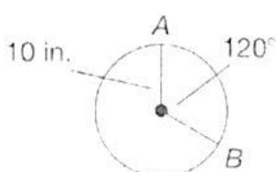


$S = \frac{m}{360} \cdot (2\pi r)$

$S = \frac{80}{360} \cdot (2\pi 10)$

$S = \frac{2}{9} \cdot 20\pi = \frac{40\pi}{9}$ or 13.96 in.

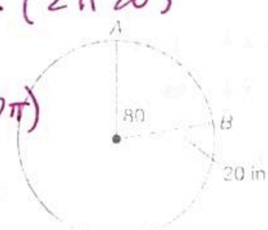
Example 2 - 3: Calculate the arc length of each circle, express your answer in terms of π.



$S = \frac{120}{360} \cdot (2\pi 10)$

$= \frac{1}{3} \cdot 20\pi$

$= \frac{20\pi}{3}$

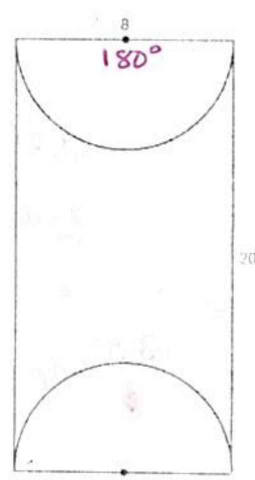


$S = \frac{80}{360} \cdot (2\pi 20)$

$= \frac{2}{9} \cdot (40\pi)$

$= \frac{80\pi}{9}$

Example 4: Two semicircle cuts were taken from the rectangular region shown.
Determine the perimeter of the shaded region. Round answer to two decimal places.



Arc length

$S = \frac{180}{360} \cdot (2\pi 4)$

$= \frac{1}{2} \cdot 8\pi$

$= 4\pi = 12.57$ (both arcs $= 12.57$)

Perimeter

$20 + 20 + 12.57 + 12.57$

$P = 65.14$

So far you have described measures of arcs and angles using degrees.

- A **radian** is the measure of a central angle whose arc length is the same as the radius of the circle.
- Radians are another unit that can be used to measure angles and arcs.

When converting *degree to radians*, multiply a degree measure by $\frac{\pi}{180°}$

When converting *radians to degrees*, multiply a radian measure by $\frac{180°}{\pi}$

θ represents the measure of the central angle in radians.

$$\theta = \frac{s}{r} \quad \begin{matrix} \rightarrow \text{arc length} \\ \rightarrow \text{radius} \end{matrix}$$

1. If $\theta = \frac{\pi}{2}$ and $r = 4$, solve for the length of the intercepted arc.

$\theta = \frac{s}{r}$

$\frac{\pi}{2} = \frac{s}{4}$

$4\pi = 2s$

$s = 2\pi$

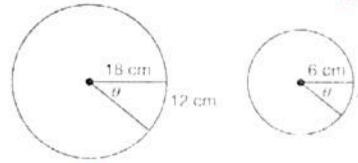
$= 6.28$

2. If $r = 2$ and the intercepted arc length is 5, what is the measure of the central angle in radians?

$\theta = \frac{s}{r}$

$\theta = \frac{5}{2} = 2.5$

3. At the same central angle, θ, if the radius is 6 cm, what is the arc length of the intercepted arc?



$\theta = \frac{s}{r}$

$\theta = \frac{12}{18} = \frac{2}{3}$ or $.67$ →

$\theta = \frac{s}{r}$

$\frac{2}{3} = \frac{s}{6}$

$12 = 3s$

$\boxed{s = 4}$

4. The measure of a central angle is 120°. The length of the radius is 20 cm.

   a. Determine the arc length using the formula: $s = \frac{m}{360°} \cdot 2\pi r$ (m)(θ) (r)

   $s = \frac{120}{360} \cdot 2\pi 20 = \frac{40\pi}{3}$ or 41.89

   $= \frac{1}{3} \cdot 40\pi$

   b. Determine the arc length using the formula: $\theta = \frac{s}{r}$

   * convert degrees to radians

   $120 \times \frac{\pi}{180} = \frac{2\pi}{3}$

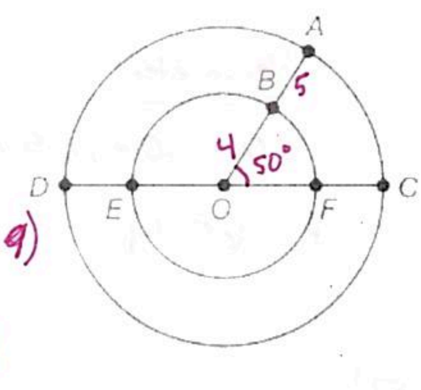
   $\frac{2\pi}{3} = \frac{s}{20}$

   $40\pi = 3s$

   $s = \frac{40\pi}{3}$ or 41.89

Check for Understanding:

- The two circles have the same center at O.
- $\overline{OB}$ is a radius of the small circle. $\overline{OA}$ is a radius of the large circle.
- $\overleftrightarrow{EF}$ is a diameter of the small circle. $\overleftrightarrow{DC}$ is a diameter of the large circle.
- $m\overline{OB} = 4in,\ m\overline{AB} = 5in,\ m\angle BOF = 50°$



$m\angle AOD$ 130°

$m\widehat{AD}$ 130°

$length\ of\ \widehat{AD}$ $\frac{13\pi}{20} = 20.42$

$s = \frac{130}{360} \cdot (2\pi 9)$

$= \frac{13}{36} \cdot (18\pi)$

$= \frac{13\pi}{20}$

$m\angle BOE$ 130°

$m\widehat{BE}$ 130°

$length\ of\ \widehat{BE}$ $\frac{13\pi}{45} = 9.08$

$s = \frac{130}{360} \cdot (2\pi 4)$

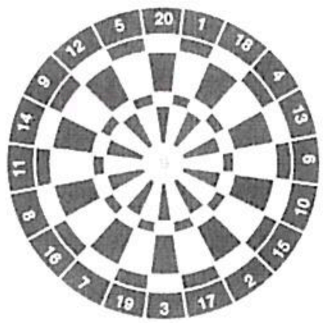
$= \frac{13}{36} \cdot (8\pi)$

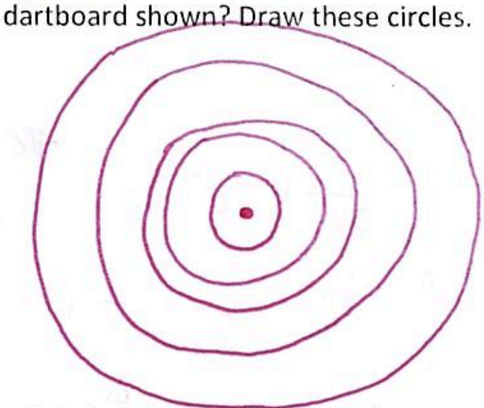
12.3:

Sectors and Segments of a Circle

- **Concentric circles**: circles that share the same center point.

How many concentric circles do you see in the dartboard shown? Draw these circles.

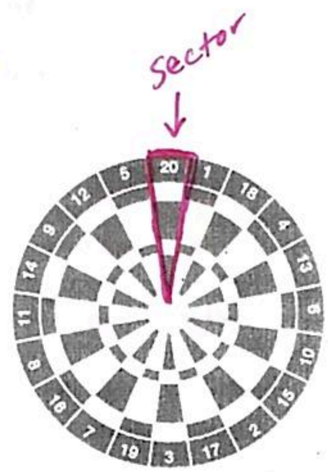




The diameter of the outermost circle is 170mm. Calculate its area in terms of π.

$A = \pi r^2$ $\quad A = \pi 85^2$ $\quad A = 7225\pi$

- **Sector of a circle**: a region of the circle bounded by two radii and the included arc.



How many congruent sectors are contained in the outermost circle of the dartboard?

20

What is the measure of the central angle and intercepted arc formed by each sector?

$20 \cong$ sectors $\rightarrow$ $360 \div 20 = 18°$

What is the ratio of the length of each intercepted arc to the circumference?

$\frac{1}{20}$

What is the ratio of the area of each sector to the area of the circle?

$\frac{1}{20}$

There is a proportional relationship between the area of a sector of a circle, A, and the area of a circle. To determine the area of a sector, A, you multiply the area of the circle by a fraction that represents the portion of the area determined by the central angle measure, m.

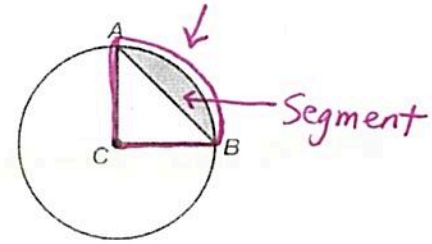
$$\frac{A}{area\ of\ circle} = \frac{m}{360°}$$

$$A = \frac{m}{360°} \cdot area\ of\ circle$$

$$A = \frac{m}{360°} \cdot (\pi r^2)$$

$A =$	area of a sector
$m =$	central ∡
$r =$	radius

○ **Segment of a circle**: a region of the circle bounded by a chord and the included arc.



If we know how to find the area of a sector,

how could we find the area of the segement of a circle?

area of Sector − area of triangle

Example 1: If the length of the radius of circle C is 8cm and $m\angle ACB = 90°$, what is the area of the shaded segment of the circle in terms of π?

$A = \frac{m}{360} \cdot \pi r^2$ (sector) $\quad A = \frac{90}{360} \cdot \pi 8^2$

$A = 16\pi$

$A(\Delta) = \frac{bh}{2}$

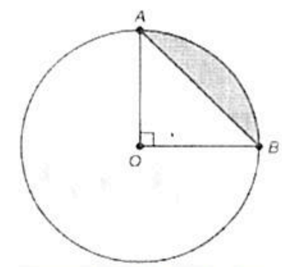
$= \frac{8 \cdot 8}{2}$

$= 32$

$A(\text{segment}) = 16\pi - 32$

Example 2: The area of the segment shown is $\boxed{9\pi} - \boxed{18\,ft^2}$, what is the radius of circle O?

Sector $\quad \Delta$

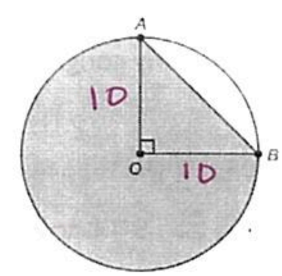


$A(\Delta) = \frac{bh}{2}$

$18 = \frac{r \cdot r}{2}$

$36 = r^2 \quad r = 6$

Example 3: The length of the radius is 10in. What is the area of the shaded region of circle O?



$A = \frac{90}{360} \cdot \pi 10^2$

$A = 25\pi$

$\Delta = \frac{10 \cdot 10}{2}$

$\Delta = 50$

Segment $= 25\pi - 50$

$= 28.54$

$O = \pi 10^2$

$= 100\pi$

$= 314.16$

− segment

Shaded area $= 285.62\ in^2$

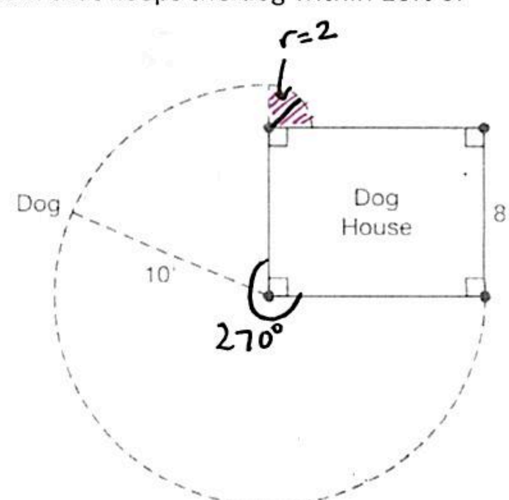
Check for Understanding:

A dog is tethered to the corner of his rectangular dog house with a 10ft chain that keeps the dog within 10ft of the corner.

1. Determine the area of the sector determined by the 10ft radius.

$A = \frac{270}{360} \cdot \pi 10^2 \quad = \frac{300\pi}{4}$

$= 75\pi$



2. What is the radius of the smaller circle?

2ft.

3. Determine the total area or space the dog can move around in.

$A = \frac{90}{360} \cdot \pi 2^2$

$= \frac{4\pi}{4} = \pi$

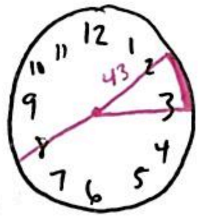
$75\pi + \pi = 76\pi$ or $\boxed{238.76\ ft^2}$

12.4: Circle Problems

Problem 1: Abraj Al Bait Towers clock in Mecca, Saudi Arabia, has a clock face with a diameter of 43 meters.

a. Determine the length of an arc connecting any two numbers on the clock face.

12 $\cong$ Sectors and arcs



$360 \div 12 = 30°$

$$S = \frac{m}{360} \cdot 2\pi r$$
$$= \frac{30}{360} \cdot 2\pi\, 21.5$$
$$= \frac{1}{12} \cdot 43\pi = \frac{43\pi}{12} \text{ or } \boxed{11.26 \text{ m}}$$

b. Determine the area of the sector formed by the minute hand and the hour hand when the time in 1:00.

$$A = \frac{m}{360} \cdot \pi r^2$$
$$= \frac{30}{360} \cdot \pi\, 21.5^2$$
$$= \frac{1}{12} \cdot 462.25\pi$$
$$= \frac{462.25\pi}{12} \text{ or } \boxed{121.02 \text{ m}^2}$$

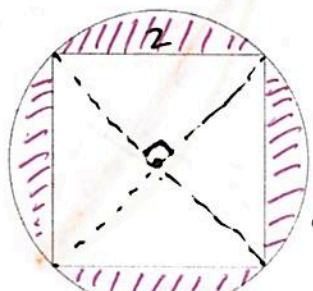
Problem 2: A square is inscribed in a circle. The length of each side of the square in 2cm. Determine the area of the shaded region.

Diagonal of □

$2^2 + 2^2 = x^2$

diameter $= \sqrt{8}$

radius $= \frac{\sqrt{8}}{2} = \sqrt{2}$



shaded = Circle − square

$\pi r^2 - bh$

$\pi \sqrt{2}^2 - 2 \cdot 2$

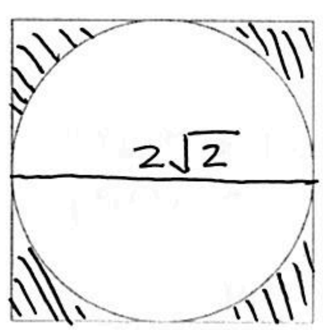
$2\pi - 4$

$\boxed{2.28 \text{ cm}^2}$

Problem 3: A circle is inscribed in a square. The length of the diameter or the circle is $2\sqrt{2}$. Determine the area of the shaded region.

radius $= \frac{2\sqrt{2}}{2}$

$= \sqrt{2}$



Shaded = Square − circle

$bh - \pi r^2$

$2\sqrt{2} \cdot 2\sqrt{2} - \pi \sqrt{2}^2$

$8 - 2\pi$

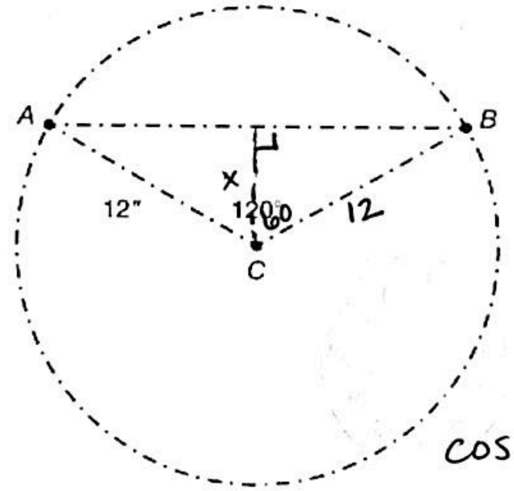
$\boxed{1.72}$

Problem 4:

$AC = 12\ inches$

$m\angle ACB = 120°$

Determine the area of segment AB.



Segment = Sector − triangle

150.8 − 62.34

$\boxed{88.46\ in^2}$

Sector $A = \frac{m}{360} \cdot \pi r^2$

$= \frac{120}{360} \cdot \pi 12^2$

$= 150.8\ in^2$

$\Delta = \frac{bh}{2}$

$= \frac{20.78(6)}{2}$

$= 62.34\ in^2$

$\cos 60 = \frac{x}{12}$

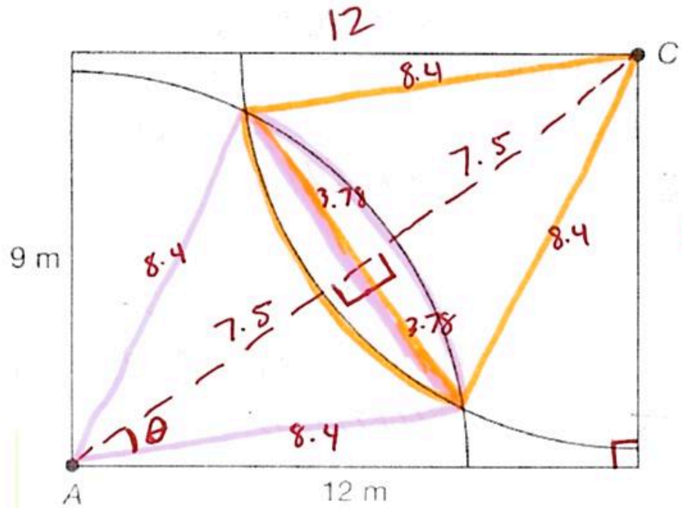
$x = 6$

$base^2 = 12^2 - 6^2$

$base = 10.39(2) = 20.78$

Problem 5: Anglea is installing sprinklers in ther front lawn. She has a rectangular lawn that measures 9 meters by 12 meters. She plans to install the sprinklers in opposite corners as shown by points A and C. Each sprinkler roatates in quarter circles and sprays water 8.4 meters.

☆Challenge Problem



Diagonal of □

$12^2 + 9^2 = x^2$

$x = 15$

height of $\Delta = 7.5$

a. How much of the lawn will be watered by both sprinklers?

Segment = sector − Δ

= 32.96 − 28.35

= 4.61 (2) = $\boxed{9.22\ m^2 \text{ overlap}}$

$\Delta \rightarrow 8.4^2 - 7.5^2 = base^2$

base = 3.78

Area Δ $= \frac{bh}{2} = \frac{7.56 \cdot 7.5}{2}$

$= 28.35\ m^2$

b. How much of the lawn will not be watered by the sprinklers?

□ − $2 \cdot \frac{1}{4}$ circles + overlap

108 − 110.84 + 9.22

$\boxed{6.38\ m^2 \text{ not watered}}$

Sector → central ∡ = $\cos\theta = \frac{7.5}{8.4}$

$\theta = \cos^{-1}\left(\frac{7.5}{8.4}\right)$

$\theta = 26.77°$

central ∡ = 53.53°

$A = \frac{53.53}{360} \cdot \pi 8.4^2$

$A = 32.96\ m^2$