A Little Dash of Logic

Foundations for Proof

2.1

Coach C's Notes

LEARNING GOALS

In this lesson, you will:

- Define inductive and deductive reasoning.
- Identify methods of reasoning.
- Compare and contrast methods of reasoning.
- Create examples using inductive and deductive reasoning.
- Identify the hypothesis and conclusion of a conditional statement.
- Explore the truth values of conditional statements.
- Use a truth table.

KEY TERMS

- induction
- deduction
- counterexample
- conditional statement
- propositional form
- propositional variables
- hypothesis
- conclusion
- truth value
- truth table

One of the most famous literary detectives is Sherlock Holmes. Created by author Sir Arthur Conan Doyle, Sherlock Holmes first appeared in print in 1887 in the novel *A Study in Scarlet*. The character has gone on to appear in four novels, 56 short stories, and over 200 films. The Guinness Book of World Records lists Holmes as the most portrayed movie character, with more than 70 different actors playing the part.

Holmes is most famous for his keen powers of observation and logical reasoning, which always helped him solve the case. In many literary and film adaptations, Holmes is known to remark, "Elementary, my dear Watson," after explaining to his assistant how he solved the mystery. However, this well-known phrase doesn't actually appear in any of the stories written by Doyle. It first appeared in the 1915 novel *Psmith, Journalist* by P.G. Wodehouse and also appeared at the end of the 1929 film *The Return of Sherlock Holmes*. Regardless, this phrase will probably always be associated with the famous detective.

PROBLEM 1 How Do You Figure?

1. Emma considered the following statements.

   - $4^2 = 4 \times 4$
   - Nine cubed is equal to nine times nine times nine.
   - 10 to the fourth power is equal to four factors of 10 multiplied together.

   Emma concluded that raising a number to a power is the same as multiplying the number as many times as indicated by the exponent. How did Emma reach this conclusion?

2. Ricky read that raising a number to a power is the same as multiplying that number as many times as indicated by the exponent. He had to determine seven to the fourth power using a calculator. So, he entered $7 \times 7 \times 7 \times 7$. How did Ricky reach this conclusion?

3. Compare Emma's reasoning to Ricky's reasoning.

4. Jennifer is a writing consultant. She is paid \$900 for a ten-hour job and \$1980 for a twenty-two-hour job.

   a. How much does Jennifer charge per hour?

   $$\frac{\$900}{10 \text{ hrs}} = \frac{\$90}{\text{hour}} \qquad \frac{\$1980}{22 \text{ hrs}} = \frac{\$90}{\text{hour}}$$

   b. To answer Question 4, part (a), did you start with a general rule and make a conclusion, or did you start with specific information and create a general rule?

   Started with specific info. created general rule.

5. Your friend Aaron tutors elementary school students. He tells you that the job pays \$8.25 per hour.

   a. How much does Aaron earn from working 4 hours?

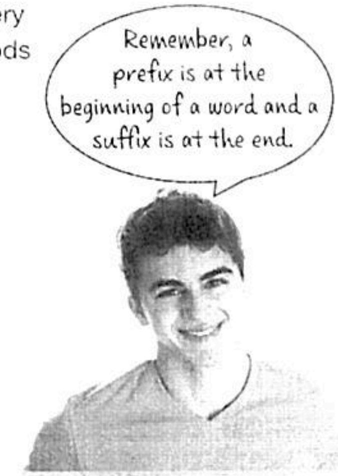
   $8.25 \times 4 = \$33$

   b. To answer Question 5, part (a), did you start with a general rule and make a conclusion, or did you start with specific information and create a general rule?

   Started with general rule, made a conclusion

PROBLEM 2 Is This English Class or Algebra?

The ability to use information to reason and make conclusions is very important in life and in mathematics. There are two common methods of reasoning. You can construct the name for each method of reasoning using your knowledge of prefixes, root words, and suffixes.



Word Fragment	Prefix, Root Word, or Suffix	Meaning
in-	Prefix	*toward* or *up to*
de-	Prefix	*down from*
-duc-	Root Word	*to lead* and often *to think*, from the Latin word *duco*
-tion	Suffix	*the act of*

1. Form a word that means "the act of thinking down from."

   Deduction

2. Form a word that means "the act of thinking toward or up to."

   Induction

general
↑
Specific

Induction is reasoning that uses specific examples to make a conclusion. Sometimes you will make generalizations about observations or patterns and apply these generalizations to new or unfamiliar situations. For example, you may notice that when you don't study for a test, your grade is lower than when you do study for a test. You apply what you learned from these observations to the next test you take.

general
↓
Specific

Deduction is reasoning that uses a general rule to make a conclusion. For example, you may learn the rule for which direction to turn a screwdriver: "righty tighty, lefty loosey." If you want to remove a screw, you apply the rule and turn the screwdriver counterclockwise.

3. Consider the reasoning used by Emma, Ricky, Jennifer, and Aaron in Problem 1.

   a. Who used inductive reasoning?

   Jennifer

   b. Who used deductive reasoning?

   Aaron

PROBLEM 3 Coming to Conclusions

A problem situation can provide you with a great deal of information that you can use to make conclusions. It is important to identify specific and general information in a problem situation to reach appropriate conclusions. Some information may be irrelevant to reach the appropriate conclusion.

Ms. Ross teaches an Economics class every day from 1:00 PM to 2:15 PM. Students' final grade is determined by class participation, homework, quizzes, and tests. She noticed that Andrew has not turned in his homework 3 days this week. She is concerned that Andrew's grade will fall if he does not turn in his homework.

Irrelevant information:
Ms. Ross teaches an Economics class every day from 1:00 PM to 2:15 PM.

General information:
Students' final grade is determined by class participation, homework, quizzes, and tests.

Specific information:
Andrew has not turned in his homework 3 days this week.

Conclusion:
Andrew's grade will fall if he does not turn in his homework.

1. Did Ms. Ross use induction or deduction to make this conclusion? Explain your answer.

Deduction - used rule about how grades were calculated to make a conclusion

2. Conner read an article that claimed that tobacco use greatly increases the risk of getting cancer. He then noticed that his neighbor Matilda smokes. Conner is concerned that Matilda has a high risk of getting cancer.

a. Which information is specific and which information is general in this problem situation?

general – tobacco increases risk of getting cancer

Specific – Matilda smokes

b. What is the conclusion in this problem?

Matilda has high risk of getting cancer.

c. Did Conner use inductive or deductive reasoning to make the conclusion? Explain your reasoning.

Deductive – general info to make conclusion.

d. Is Conner's conclusion correct? Explain your reasoning.

Yes.
She may not get cancer
but smoking increases her risk.

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3. Molly returned from a trip to England and tells you, "It rains every day in England!" She explains that it rained each of the five days she was there.

Skip

a. Which information is specific and which information is general in this problem situation?

2

b. What is the conclusion in this problem?

c. Did Molly use inductive or deductive reasoning to make the conclusion? Explain your answer.

d. Is Molly's conclusion correct? Explain your reasoning.

4. Dontrell takes detailed notes in history class and math class. His classmate Trang will miss biology class tomorrow to attend a field trip. Trang's biology teacher asks him if he knows someone who always takes detailed notes. Trang tells his biology teacher that Dontrell takes detailed notes. Trang's biology teacher suggests that Trang should borrow Dontrell's notes because he concludes that Dontrell's notes will be detailed.

Skip

a. What conclusion did Trang make? What information supports this conclusion?

2

b. What type of reasoning did Trang use? Explain your reasoning.

c. What conclusion did the biology teacher make? What information supports this conclusion?

d. What type of reasoning did the biology teacher use? Explain your reasoning.

e. Will Trang's conclusion always be true? Will the biology teacher's conclusion always be true? Explain your reasoning.

Skip

5. The first four numbers in a sequence are 4, 15, 26, and 37.

   a. What is the next number in the sequence? How did you calculate the next number?

   b. Describe how you used both induction and deduction, and what order you used these reasonings to make your conclusion.

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6. The first three numbers in a sequence are 1, 4, 9 . . . Marie and Jose both determined that the fourth number in the sequence is 16. Marie's rule involved multiplication whereas Jose's rule involved addition.

   a. What types of reasoning did Marie and Jose use to determine the fourth number in the sequence?

   b. What rule did Marie use to determine the fourth number in the sequence?

   c. What rule did Jose use to determine the fourth number in the sequence?

   d. Who used the correct rule? Explain your reasoning.

PROBLEM 4 Why Is This False?

There are two reasons why a conclusion may be false. Either the assumed information is false, or the argument is not valid.

1. Derek tells his little brother that it will not rain for the next 30 days because he "knows everything." Why is this conclusion false?

   Because the assumed information is false. No one knows everything.

2. Two lines are not parallel, so the lines must intersect. Why is this conclusion false?

   Because the argument is not valid. They could be skew lines.

3. Write an example of a conclusion that is false because the assumed information is false.

   All boys like football. John is a boy. So, John likes football.

   (some boys don't like football)

4. Write an example of a conclusion that is false because the argument is not valid.

   Kylie doesn't like green fruit. Apples are green so Kylie doesn't like apples.

   (Apples can also ~~be~~ red and yellow)

To show that a statement is false, you can provide a *counterexample*. A **counterexample** is a specific example that shows that a general statement is not true.

5. Provide a counterexample for each of these statements to demonstrate that they are not true.

   a. All prime numbers are odd.

   2 is a prime number that is even.

   b. The sum of the measures of two acute angles is always greater than 90°.

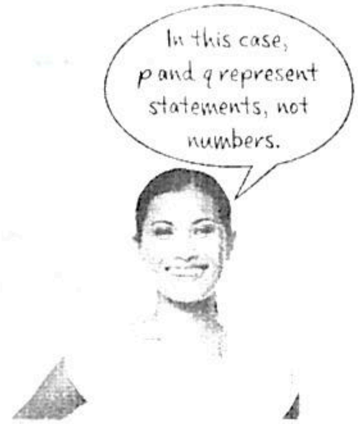
   Two acute angles that measure 35° each

   -Their sum would be 70°

PROBLEM 5 You Can't Handle the Truth Value

A **conditional statement** is a statement that can be written in the form "If p, then q." This form is the **propositional form** of a conditional statement. It can also be written using symbols as $p \to q$, which is read as "p implies q." The variables p and q are **propositional variables**. The **hypothesis** of a conditional statement is the variable p. The **conclusion** of a conditional statement is the variable q.

The **truth value** of a conditional statement is whether the statement is true or false. If a conditional statement could be true, then the truth value of the statement is considered true. The truth value of a conditional statement is either true or false, but not both.



You can identify the hypothesis and conclusion from a conditional statement.

Conditional Statement
If $x^2 = 36$, then $x = 6$ or $x = -6$.

Hypothesis of the Conditional Statement
$x^2 = 36$

Conclusion of the Conditional Statement
$x = 6$ or $x = -6$.

Consider the conditional statement: If the measure of an angle is 32°, then the angle is acute.

1. What is the hypothesis p?

   The measure of an angle is 32°

2. What is the conclusion q?

   The angle is acute.

3. If p is true and q is true, then the truth value of a conditional statement is true.

   a. What does the phrase "If p is true" mean in terms of the conditional statement?

   that the measure of the angle is 32°

   b. What does the phrase "If q is true" mean in terms of the conditional statement?

   that the angle is acute.

   c. Explain why the truth value of the conditional statement is true if both p and q are true.

   if it's less than 90° then it is acute and the angle is 32° which is less than 90°.

4. If p is true and q is false, then the truth value of a conditional statement is false.

   a. What does the phrase "If p is true" mean in terms of the conditional statement?

   that the measure of the angle is 32°

   b. What does the phrase "If q is false" mean in terms of the conditional statement?

   that the angle is not acute

   c. Explain why the truth value of the conditional statement is false if p is true and q is false.

   if an acute angle is less than 90°
   then the statement an angle that is 32° is not acute is a false statement.

5. If p is false and q is true, then the truth value of a conditional statement is true.

   a. What does the phrase "If p is false" mean in terms of the conditional statement?

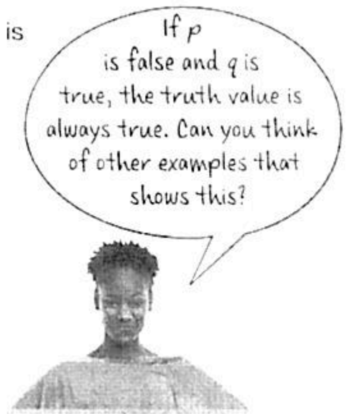
   measure of angle is not 32°

   b. What does the phrase "If q is true" mean in terms of the conditional statement?

   the angle is acute.

   c. Explain why the truth value of the conditional statement is true if p is false and q is true.

   angle could still be less than 90°, still accurate statement



6. If p is false and q is false, then the truth value of a conditional statement is true.

   a. What does the phrase "If p is false" mean in terms of the conditional statement?

   angle is not 32°

   b. What does the phrase "If q is false" mean in terms of the conditional statement?

   angle is not acute

   c. Explain why the truth value of the conditional statement is true if both p and q are false.

   angle could be greate than 90°, statement can be true.

2

A **truth table** is a table that summarizes all possible truth values for a conditional statement $p \rightarrow q$. The first two columns of a truth table represent all possible truth values for the propositional variables p and q. The last column represents the truth value of the conditional statement $p \rightarrow q$.

The truth values for the conditional statement "If the measure of an angle is 32°, then the angle is acute" is shown.

The truth value of the conditional statement $p \rightarrow q$ is determined by the truth value of p and the truth value of q.

- If p is true and q is true, then $p \rightarrow q$ is true.
- If p is true and q is false, then $p \rightarrow q$ is false.
- If p is false and q is true, then $p \rightarrow q$ is true.
- If p is false and q is false, then $p \rightarrow q$ is true.

p	q	$p \rightarrow q$
the measure of an angle is 32°	**the angle is acute**	**If the measure of an angle is 32°, then the angle is acute.**
T	T	T
T	F	F
F	T	T
F	F	T

7. Consider the conditional statement: If $m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches, then $\overline{AB} \cong \overline{BC}$.

a. What is the hypothesis p?

$m\overline{AB} = 6$ inches and $m\overline{BC} = 6$ inches

b. What is the conclusion q?

$\overline{AB} \cong \overline{BC}$

c. If both p and q are true, what does that mean? What is the truth value of the conditional statement if both p and q are true?

if both segments equal to 6 inches then two segments are congruent.

* Truth value is true.

not congruent

d. If p is true and q is false, what does that mean? What is the truth value of the conditional statement if p is true and q is false?

If $m\overline{AB} = 6$ and $m\overline{BC} = 6$ then they are not congruent

* Truth value is false

e. If p is false and q is true, what does that mean? What is the truth value of the conditional statement if p is false and q is true?

If $m\overline{AB}$ and $m\overline{BC}$ is not equal to 6 then they are congruent

* Truth value is true
(they could be = to a different value)

2

f. If both p and q are false, what does that mean? What is the truth value of the conditional statement if both p and q are false?

If $m\overline{AB}$ and $m\overline{BC}$ are not = 6 then they are not congruent

* Truth value is true
(they could be other values)

g. Summarize your answers to parts (a) through (f) by completing a truth table for the conditional statement.

p	q	$p \rightarrow q$
$m\overline{AB} = 6$ in $m\overline{BC} = 6$ in	$\overline{AB} \cong \overline{BC}$	If $m\overline{AB} = 6$ and $m\overline{BC} = 6$ then $\overline{AB} \cong \overline{BC}$
T	T	T
T	F	F
F	T	T
F	F	T

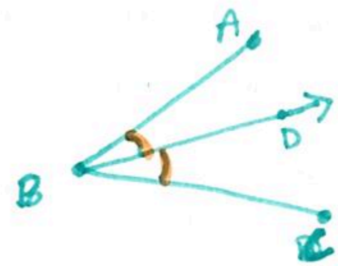
PROBLEM 6 Rewriting Conditional Statements

For each conditional statement, draw a diagram and then write the hypothesis as the "Given" and the conclusion as the "Prove."

1. If $\overrightarrow{BD}$ bisects $\angle ABC$, then $\angle ABD \cong \angle CBD$.

Given: $\overrightarrow{BD}$ bisects $\angle ABC$

Prove: $\angle ABD \cong \angle CBD$



2. $\overline{AM} \cong \overline{MB}$, if M is the midpoint of $\overline{AB}$.

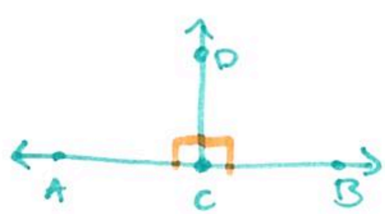
Given: M is midpoint of $\overline{AB}$

Prove: $\overline{AM} \cong \overline{MB}$

3. If $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ at point C, then $\angle ACD$ is a right angle and $\angle BCD$ is a right angle.

Given: $\overleftrightarrow{AB} \perp \overrightarrow{CD}$ at C

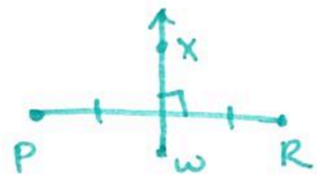
Prove: $\angle ACD$ and $\angle BCD$ are right angles



4. $\overleftrightarrow{WX}$ is the perpendicular bisector of $\overline{PR}$, if $\overleftrightarrow{WX} \perp \overline{PR}$ and $\overleftrightarrow{WX}$ bisects $\overline{PR}$.

Given: $\overleftrightarrow{WX} \perp \overline{PR}$ and $\overrightarrow{WX}$ bisects $\overline{PR}$

Prove: $\overrightarrow{WX}$ is perpendicular bisector of $\overline{PR}$



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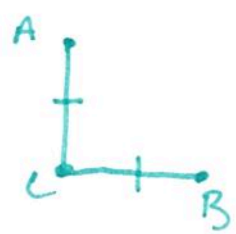
5. Mr. David wrote the following information on the board.

If $\overline{AC} \cong \overline{BC}$, then C is the midpoint of $\overline{AB}$.

He asked his students to discuss the truth of this conditional statement.

Susan said she believed the statement to be true in all situations. Marcus disagreed with Susan and said that the statement was not true all of the time.

What is Marcus thinking and who is correct?



The two segments could be congruent without C being a midpoint.